The Effect of Time, Trend, Volatility, and Leverage on Relative Returns

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Executive Summary

- This study investigates the effect of time, return trend, volatility, leverage ratio, and the rebalancing period on leveraged Exchange Traded Funds' (ETFs) returns relative to the underlying index.

- For investors with holding periods of longer than a month, leveraged ETFs should only be considered when the market is expected to trend during times of low volatility. The greater the leverage, the more significant this condition needs to be. Due to both a base and compounding issue, short-levered ETFs have even faster degradation in returns relative to the leverage ratio multiplied by the underlying index return.

- The greater the volatility of the underlying index, the greater the decline in long-term levered ETF returns relative to the index return multiplied by the fund's stated beta (daily leverage ratio). If volatility is great enough, long-term returns from holding leveraged ETFs can be in the opposite direction of the daily leverage ratio. In essence, a long- (short-) leveraged fund can lose money despite the underlying index being up (down).

- New ETFs using monthly, as opposed to daily, rebalancing are currently going through the SEC approval process. One fund company already uses monthly rebalancing. These funds do help reduce the degradation of leveraged returns over holding periods of less than one year, but the difference between daily and monthly rebalancing becomes almost indistinct with holding periods of a year or more.

- There is also an interim investor problem for leveraged funds that use monthly rebalancing. The result is that actual leverage can be much greater or less than the initial leverage of the fund, depending on the underlying index value relative to when the fund initially rebalanced.

Background

More than $30 billion in assets has poured into leveraged ETFs in the last three years, despite the dramatic decline in broad market equity indices in 2008. While other funds are seeing redemptions, leveraged ETFs continue to grow. The typical objective statement for these increasingly popular funds generally reads along these lines: "The fund seeks daily investment results, before fees and expenses, of X% of the price performance of the underlying benchmark index." Currently, the X% part ranges from -3.0x to 3.0x and the underlying index ranges from the S&P 500® Index to the MSCI Emerging Markets Index and beyond. Those funds with a positive multiple are referred to as "bull" funds and those with a negative multiple are referred to as "bear" funds. For example, a bullish investor who holds a fund with a multiple of 3.0x the S&P 500 Index expects to earn a daily return of 3%[1] if the S&P 500 earns a return of 1% on a particular day. Alternatively, the bearish holder of a -3.0x fund expects to lose 3%[1] given the same daily S&P 500 performance. The opposite would occur if the S&P 500 were to drop by 1% on a particular day: The bullish investor might expect to lose 3%[1], whereas the bearish investor would anticipate a 3%[1] gain.

The most significant feature of the above description that has led to a significant misunderstanding of these funds is the fact that these funds' returns are a multiple of index returns on a "daily" basis only[2]. The misunderstanding comes from the problem that the daily multiple cannot be extended for longer-term time horizons without introducing increasing uncertainty. In fact, due to compounding issues resulting from creating a daily multiple, virtually any realized multiple is possible for any levered fund given enough time, return trend, and volatility. This multiple can work both for and against the investor, but under most market conditions, the effects of volatility generally cause long-term return multiples to be much less than a fund's daily leverage ratio.

Trainor and Baryla[3] demonstrate why these funds are not designed for the long-term investor, but may be valuable to investors willing to make a speculative play, or for short-term trading. In actuality, the implied average holding period for many of these funds is now quite short, often less than a week, suggesting that the message has been received. However, the questions remain regarding exactly how long is too long to hold these funds, and under what conditions a longer holding period can be justified.

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[1] Minus fees and expenses.

[2] For further expositions, see Justice, 2009; Lauricella, 2009; Spence, 2009; and Zwieg, 2009 in Appendix D.

[3] See Trainor and Baryla in Appendix D for more information.

This paper examines:

- the length of time a trader should consider holding these funds under various market conditions;
- when leveraged ETFs are most likely to succeed for longer-term investors; and
- the issues associated with the new ETFs that will be using monthly, instead of daily, rebalancing.

Definitions used throughout this paper:

Realized beta:

The levered fund's cumulative realized return divided by the underlying benchmark's cumulative return over any given time frame. On a daily basis, realized beta will equal the leverage ratio of the fund.

Beta divergence:

The realized beta of the levered fund minus the daily leverage ratio.

Basic Leveraged Fund Properties

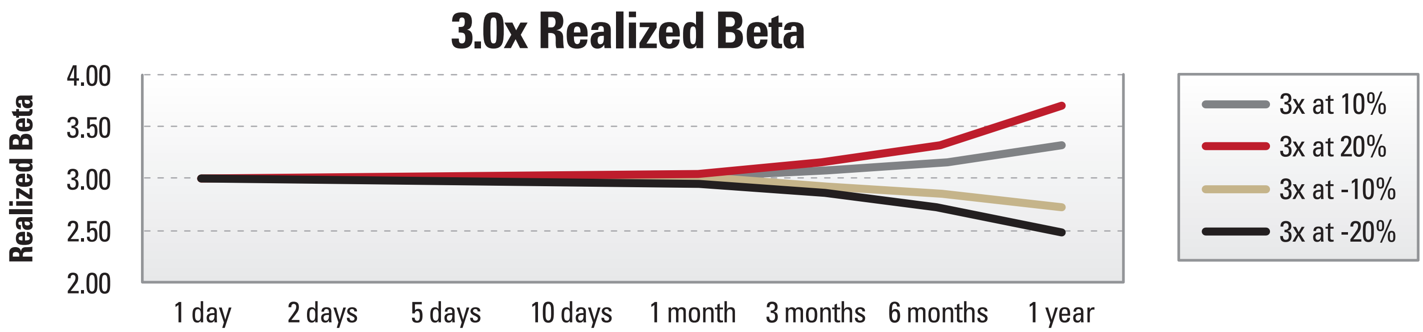
For those uninitiated with the properties of leveraged funds, three specific properties need to be considered and understood, which are detailed in the following theorems and proven mathematically in Appendix A.

Theorem 1: Assuming no volatility, the greater the trend, the greater the beta divergence. For both bullish and bearish levered funds, a positive trend will result in positive beta divergence, and a negative trend will result in negative beta divergence. These divergences will increase over time.

Theorem 1 tells us that for long-levered funds, the greater the positive trend of the underlying index, the more positive the beta divergence. Figure 1 represents this graphically and shows that long-levered funds can actually return more than the daily leverage ratio over time. For example, a 3.0x levered fund after six months will have a realized beta of 3.15, resulting in a beta divergence of 0.15 from the underlying daily beta.

Theorem 1 also tells us that if the market trends against the investor, the beta divergence will actually be negative. Specifically, this means the accumulated loss will be less than the daily leverage ratio multiplied by the cumulative return of the underlying index. As an example, if the market decreases 20% annually in a non-stochastic linear fashion, a 3.0x fund will be down less than 60% in a year. Figure 1 shows that after 1 year, the actual return for a 3.0x fund will be approximately 2.49x the underlying index's cumulative return. It is also apparent from Figure 1 that beta divergence increases in absolute value over time.

Figure 1: Realized 3.0x beta over time, assuming a non-stochastic -20% to +20% nominal annual average trend

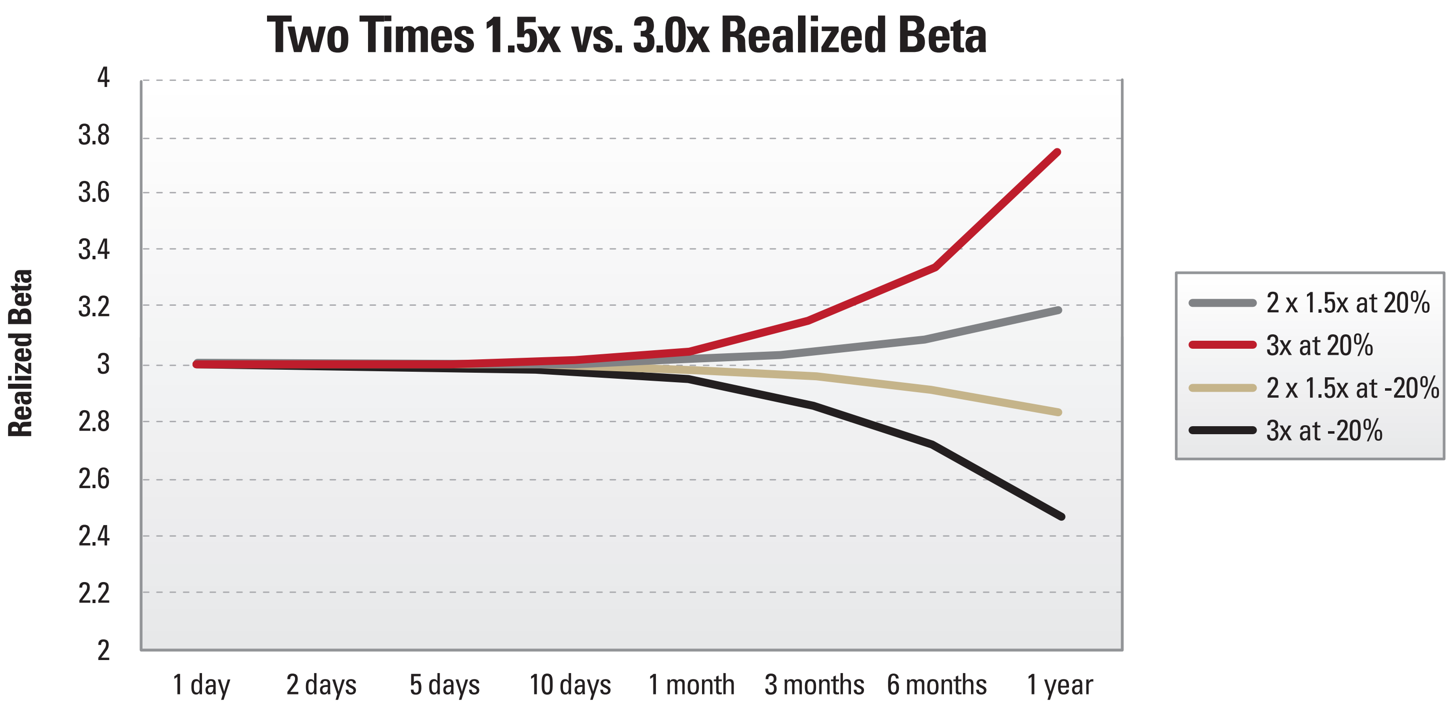


The above ideas hold for short funds as well. That is, a -3.0x fund will have a loss of less than -3.0x the underlying index when the index trends up, and will have more than a 3.0x gain when the index trends down. See “Theorem 1: Negative Beta Divergence When Market Trends Against the Investor” in Appendix C for more information. Thus, all else being equal and in conditions of very low or zero volatility, a trending market is a positive for the multi-day or long-term-levered ETF holder. Gains are magnified for a correct market call and cumulative losses are limited relative to the daily leverage ratio.

Theorem 2: The greater the leverage, the greater the trend effect.

Theorem 2 tells us that any trend effect is magnified exponentially by greater leverage. What this means is that the trend effect for a 3.0x fund will be more than double the effect from a 1.5x fund. Figure 2 demonstrates this for a 1.5x fund and a 3.0x fund over a one-year period when the underlying index has increased linearly by 20% over the year under the assumption of zero volatility.

Figure 2: Realized beta value over time, assuming a 20% and -20% nominal annual non-stochastic trend. The 1.5x beta is multiplied by 2 to demonstrate that a 3.0x fund's beta will increase relative to a 1.5x fund's beta.



For investors who are considering a 50% investment in a 3.0x fund and 50% in cash, or a 100% investment in a 1.5x fund, the effect of the trend would cause the investment in the 3.0x fund to outperform, even though the investor begins with the same initial overall leverage. Conversely, if the underlying index trended in the other direction, the 3.0x investor would actually lose less.

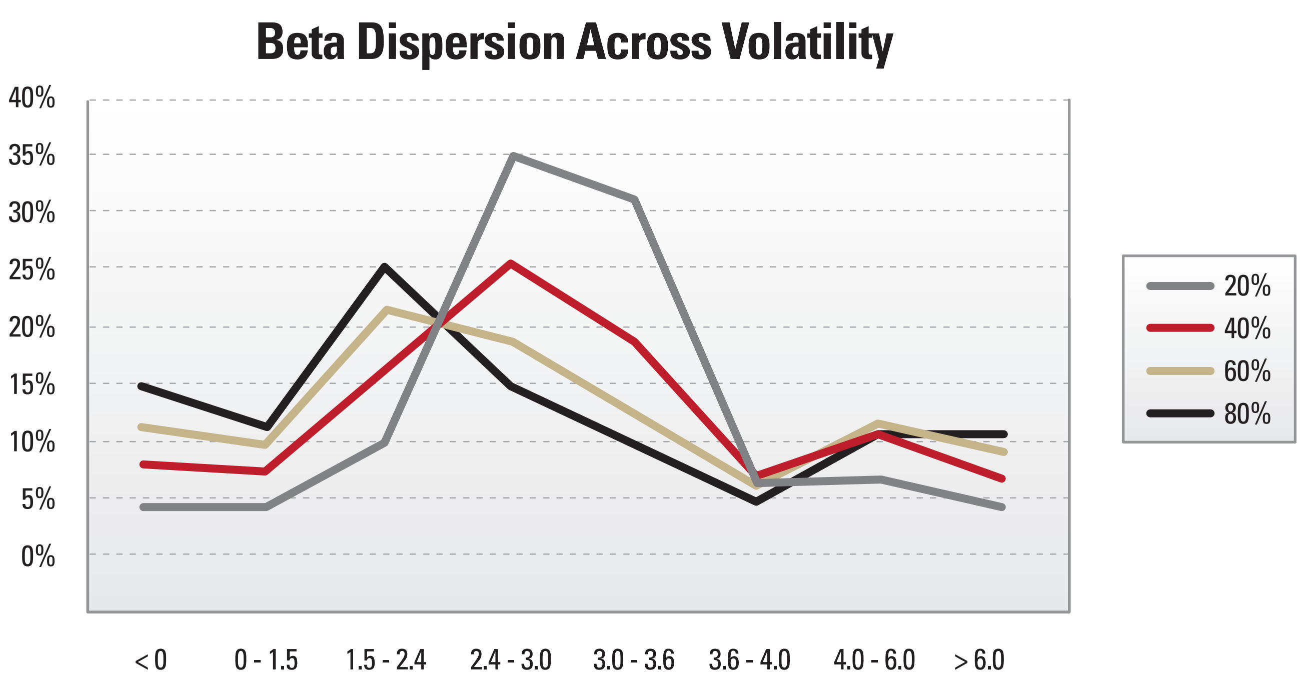
Figure 2 shows that when the trend is negative, such as -20%, the 3.0x fund will actually be down only 2.48x, while twice the 1.5x fund is down 2.86 times the underlying index. The rationale behind this is based on the daily rebalancing of the funds. As the index trends down, the losses are applied to a smaller total wealth in the 3.0x fund, as opposed to an investor who has twice as much in a 1.5x fund. It should be reiterated that this example is specific to the assumption of a non-stochastic declining trend. Under ordinary market conditions with typical day-to-day volatility, the 3.0x would not be expected to lose less than twice the 1.5x fund. The above idea holds for short funds as well. See "Theorem 2: The Effect of Trends – -1.0x vs. -3.0x Funds" in Appendix C for more information.

Theorem 3: The greater the volatility of the underlying index, the greater the decline in median beta values relative to their daily leverage ratios. In addition, the greater the leverage and the longer the time frame, the greater volatility's effect will be.

Theorem 3 states that, on average, the beta for any long-levered fund will increasingly fall below the daily leverage ratio as the volatility increases. In fact, the realized beta can turn negative, which demonstrates that a levered fund can have a cumulative return opposite of the daily leverage ratio. Unfortunately, this effect will never be in the investor's favor due to the mathematics of compounding. For instance, a long-leveraged fund can only have a negative realized beta when the market is up, and a short fund can only have a positive realized beta when the market is down. The greater the volatility, the more likely this is to happen.

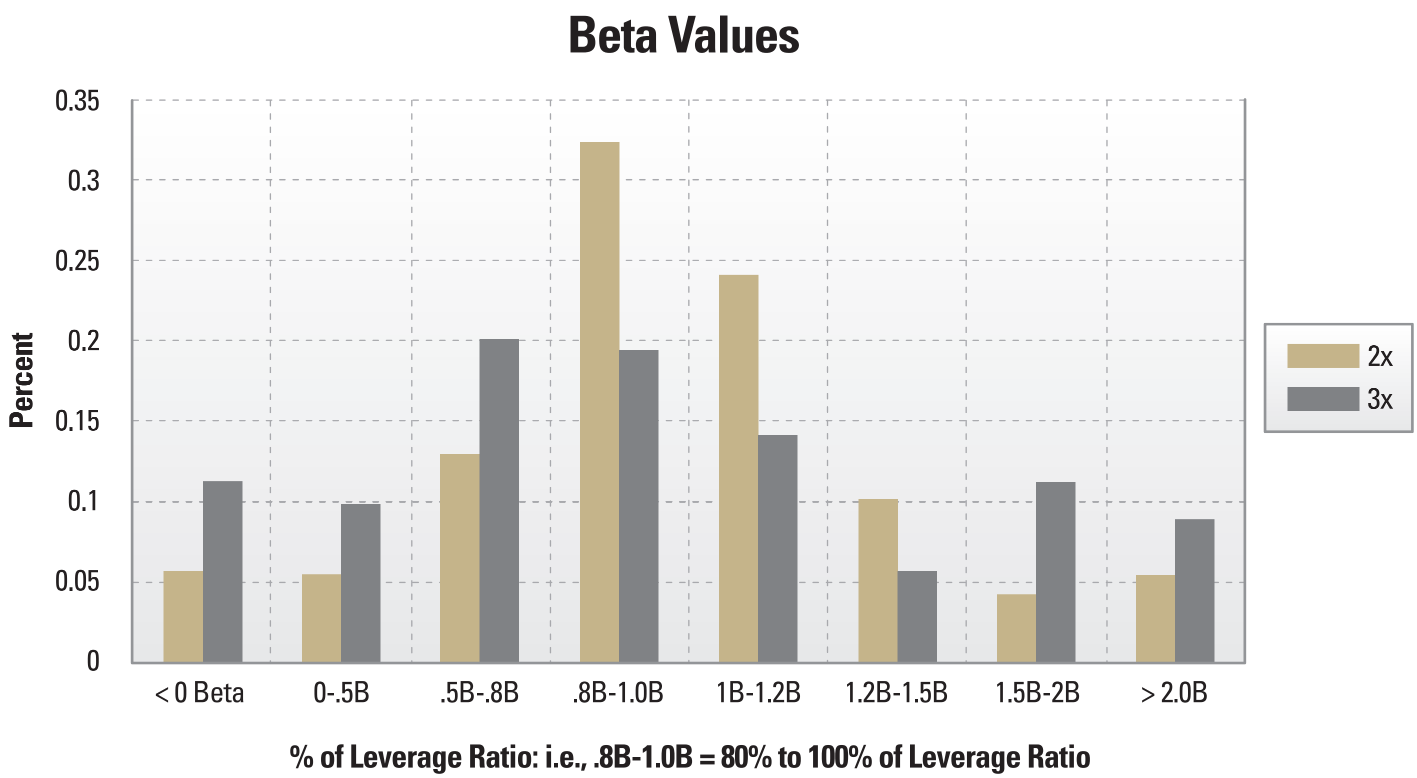
Figure 3 shows the possible beta outcomes at 3 months for a 3.0x-levered fund, assuming a 10% trend when volatility (measured as the annualized standard deviation of daily returns) is set at 20%, 40%, 60% and 80%. Results are based on 10,000 simulations using Monte Carlo methods. The 20% volatility line shows there is a 67% chance that beta will fall between 2.4 and 3.6, which is within 20% of the daily leverage ratio. There is a 4% chance the beta will be greater than 6.0. Conversely, there is a 4% chance it will be below zero. When volatility increases to 80%, Figure 3 shows there is only a 25% chance that the realized beta will be between 2.4 and 3.6 (15% between 2.4 and 3.0 plus 10% between 3.0 and 3.6). In addition, the chance of a negative beta increases to 15%. These values become more extreme over time.

Figure 3: Realized beta values at 3 months for a 3.0x fund assuming a 10% nominal annual average trend and annual volatility levels ranging from 20% to 80%



Another point of Theorem 3 is that the greater the leverage, the more extreme the results. This is graphically shown in Figure 4, which compares the difference between a 2.0x and 3.0x fund's realized beta at 6 months, when volatility is 40%. As one can see, the probability for a 3.0x fund's realized beta to be within a given percentage of the daily leverage ratio is smaller than that of a 2.0x fund. By 6 months, the 3.0x fund has less than a 35% chance that the realized beta will be within 20% of the daily leverage of 3.0. The probability of a negative beta for a 3.0x fund is over 10%, which is twice the probability for a 2.0x fund under these market conditions.

Figure 4: Probability that the realized beta value will be within a certain percentage of the daily leverage ratio



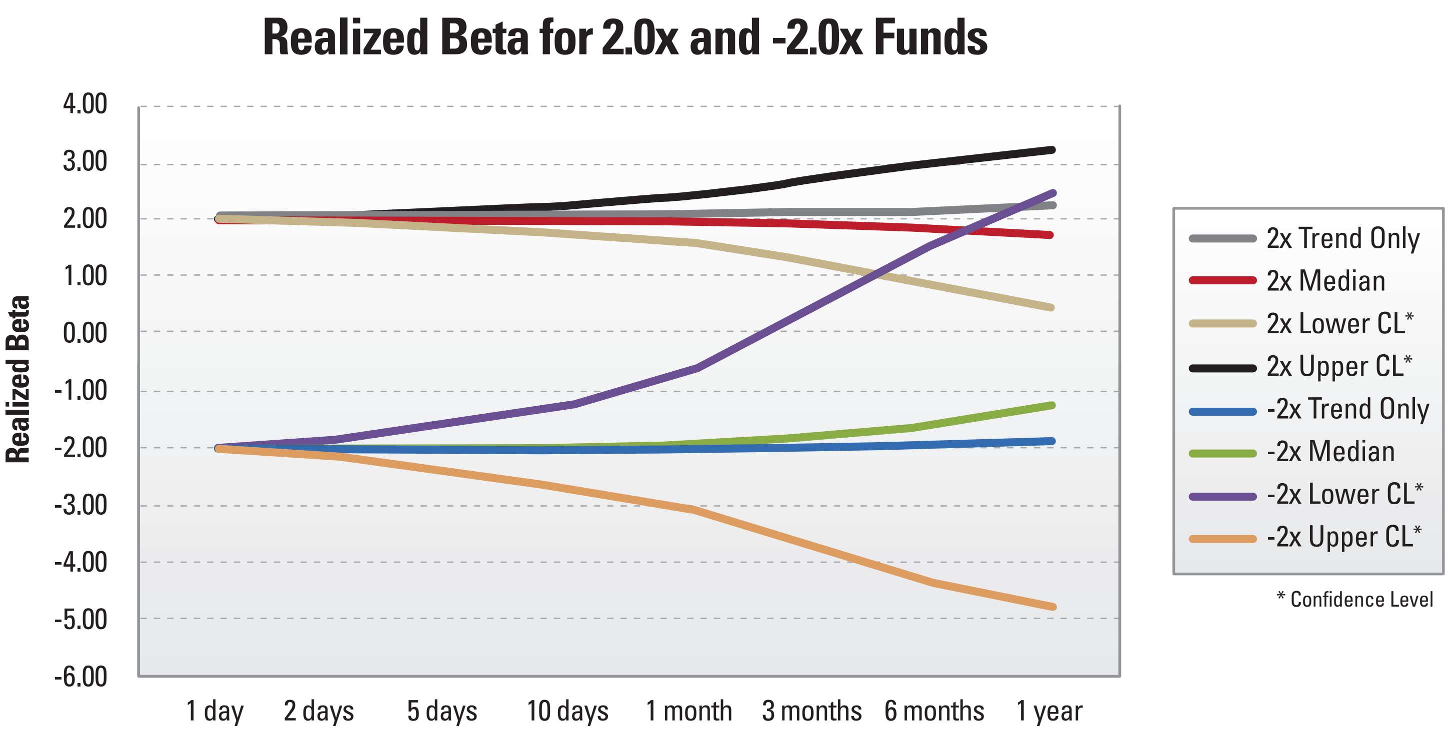
It should also be noted that volatility, the daily leverage ratio, and time are all critically linked. Figure 5 on the next page shows 2.0x and -2.0x funds with a 10% trend over time, assuming an initial 0% volatility. The initial lines represent the pure trend result discussed earlier. When 40% volatility is introduced into the equation, the expected realized beta values measured in absolute value terms fall. However, in terms of pure expectations, this beta divergence is not that significant. The major effect of volatility on these funds' relative returns is not the expected value, but the range of values that can occur.

To demonstrate this range, Figure 5 also shows the 80% confidence interval lines around the 2.0x and -2.0x funds' median values. These lines show the range in which the realized beta will fall 80% of the time under these market conditions. For the first three months, beta divergence stays within 20% of the daily leverage ratio for the 2.0x fund, but this narrow range only lasts 5 days for the -2.0x fund. This is due to the fact that even a non-leveraged short fund cannot maintain a realized beta of -1.0. (See the end of this section for more details on this point.) After a certain period of time, realized beta can diverge quite dramatically from the daily leverage ratio for any leveraged fund.

For example, after six months, the expected realized beta value for the 2.0x fund is 1.93, but this value can range from 0.86 to 2.94. Similarly, the expected realized beta value for the -2.0x fund has fallen to -1.63, but can range from 1.4 to -4.5. The greater the daily leverage, volatility, and time frame, the more dramatic this result will be.

It needs to be reiterated that a beta with a sign opposite the daily leverage ratio cannot occur when it is advantageous to the investor. Thus, the positive 1.4 beta at 6 months for the bearish -2.0x fund shows that the underlying index has a negative cumulative return over six months, but the short-leveraged fund loses 1.4 times this amount, despite being short the index.

Figure 5: Realized beta for 2.0x and -2.0x funds, assuming a 10% trend and 40% volatility



For 3.0x and -3.0x funds with a 10% trend and a 40% volatility, the results are as expected—more dramatic. Although not shown, by 1 month, the range is already 1.7 to 4.2 for the 3.0x fund and -1.8 to -4.1 for the -3.0x fund. At 6 months, the ranges are -0.4 to 5.6 and 0.2 to -5.6, respectively. Thus, the greater the daily leverage and the higher the volatility, the shorter an investor's time frame should be, at least in terms of not experiencing a high degree of beta divergence. In this scenario, the divergence begins to become severe after 10 days.

Only with high degrees of trend, low leverage, and low volatility will beta divergence be minimized. For instance, a 2.0x or -2.0x fund with 40% or -40% trend, respectively, and 20% volatility conditions has relatively minimal beta divergence even out to 1 year. (Refer to Figure C3 in Appendix C for a graph of this scenario.)

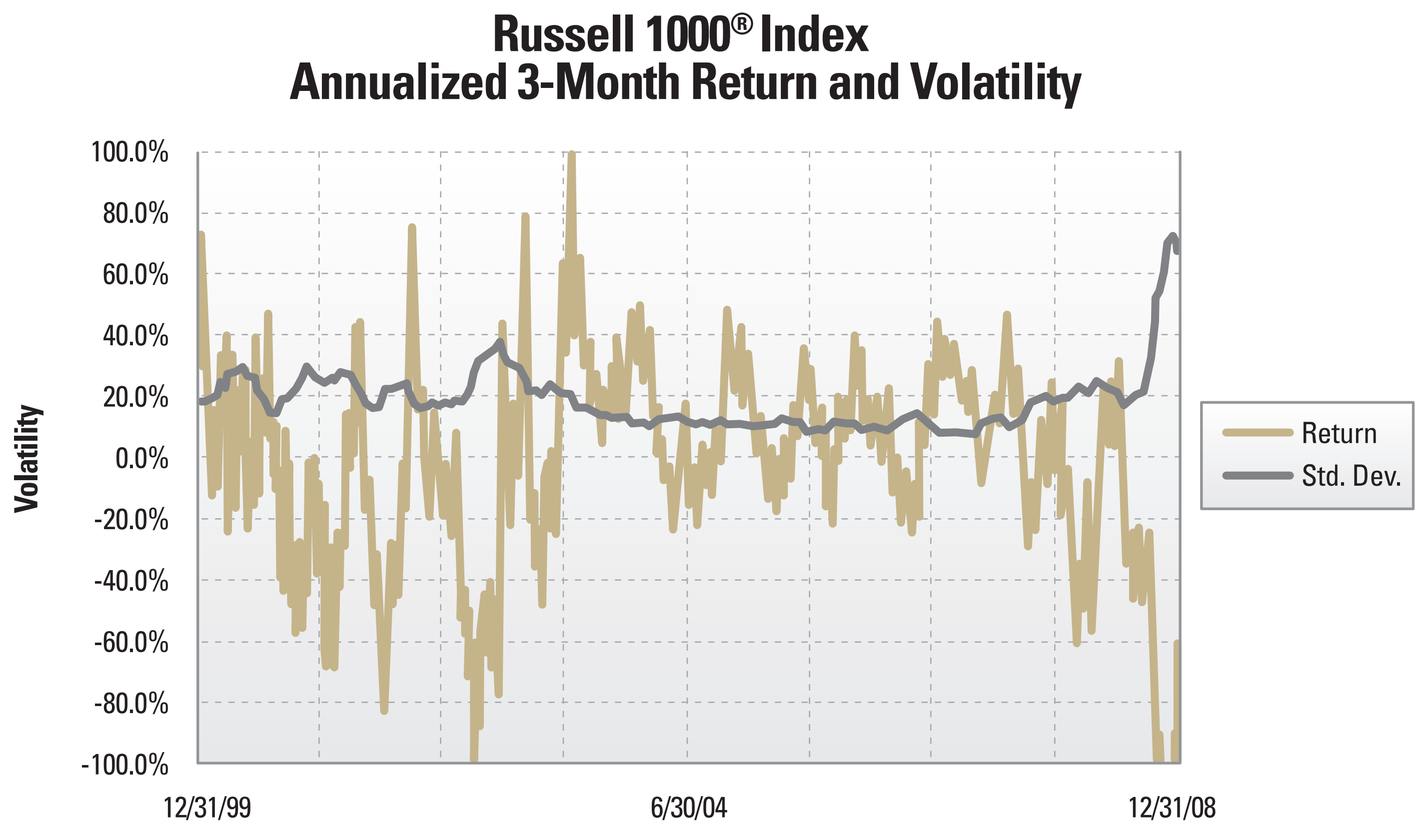
For those investors who buy short-leveraged ETFs, compounding issues are actually magnified, relative to an equally levered long fund. This is due to the fact that even a -1.0x fund cannot maintain a realized beta of -1.0 due to the base effect. Thus, a -2.0x fund has not only a compounding issue, but a base issue as well.

As an example, consider an index that increases by 10% on day 1, then falls by 20% on day 2. Starting at 100, the index increases to 110, then falls to 88. A -1.0x short fund will decrease by 10%, then increase by 20% the next day. The short fund would decrease to 90, then increase to 108. The base issue is that the 20% change on day 2 applies only to a value of 90 for the short fund and a value of 110 for the long fund. The cumulative return for the index is -12%, but the gain for the short fund is only 8%, for an effective realized beta of -0.67. A long fund does not have this problem and can have a realized beta of 1.0, whereas a short fund cannot. A -2.0x short fund not only has the base issue, but also the additional leverage issue, and thus, any increase in volatility will magnify a short fund's beta divergence relative to an equally leveraged long fund. See "Theorem 3: Beta Divergence" in Appendix C for more information.

Application to Historical Results

To give the investor some historical context about trend and volatility, three-month nominal annual returns and standard deviations based on daily returns are shown for the Russell 1000® Index from January 2000 through December 2008.

Figure 6: Russell 1000 Index annualized: rolling 3-month return and standard deviation based on daily data



As one can see, volatility has been relatively stable over this time frame, but hit a peak of over 70% in November through December of 2008. Over this eight-year time period, volatility averaged 18.2% and had a minimum of 7% in late 2006. Historically, the annual standard deviation of returns has been approximately 20% over the last 80 years. Figure 6, on the previous page, also shows that the annualized three-month trend has been much more volatile, however, with some strong periods of trend, both in negative and positive directions, ranging from -169% to 99%.

Thus, there have been many sub-periods of three months and longer which have experienced strong trend coupled with low volatility. These conditions are ideal even for the multi-day or multi-month investor in leveraged funds. Conversely, there are also periods of high volatility, such as the second half of 2002, and more recently, the last half of 2008, which continued well into 2009. In periods such as these, the negative effects of compounding are exacerbated and investing in leveraged funds with a time horizon of more than 10 to 30 days should be approached with extreme caution. To learn more about the effects that different types of historical periods can have on leveraged funds, refer to "Historical Periods & Their Effects on Leveraged Funds" in Appendix C.

To further investigate this and attain an idea of how many trending periods are favorable for an investor with a three-month holding period, Figure 7 shows the time period from 2000 to 2008 in which the realized beta is 100% or more of the daily leverage ratio based on daily rolling three-month periods for a 3.0x fund. The standard deviation and three-month return are superimposed.

As Figure 7 shows, there are several periods in which a three-month or longer holding period had favorable compounding results. However, Figure 7 also shows that the trend has to be fairly significant—almost always more than 5%—and the standard deviation has to have fallen or remained approximately below 15%. Until the annualized standard deviation fell below 15% between mid-2003 and mid-2007, there are virtually no favorable compounding periods. Since mid-2007, there have been none for long funds. This is illustrated in "-3.0x Fund Trends: Jan. 2000-Dec. 2008" in Appendix C.

Figure 7: 3-month periods in which realized beta is greater than 100% of daily leverage ratio given the market trend is up for a 3.0x fund. Time periods are delineated by columns equal to 3.0 when this occurs.

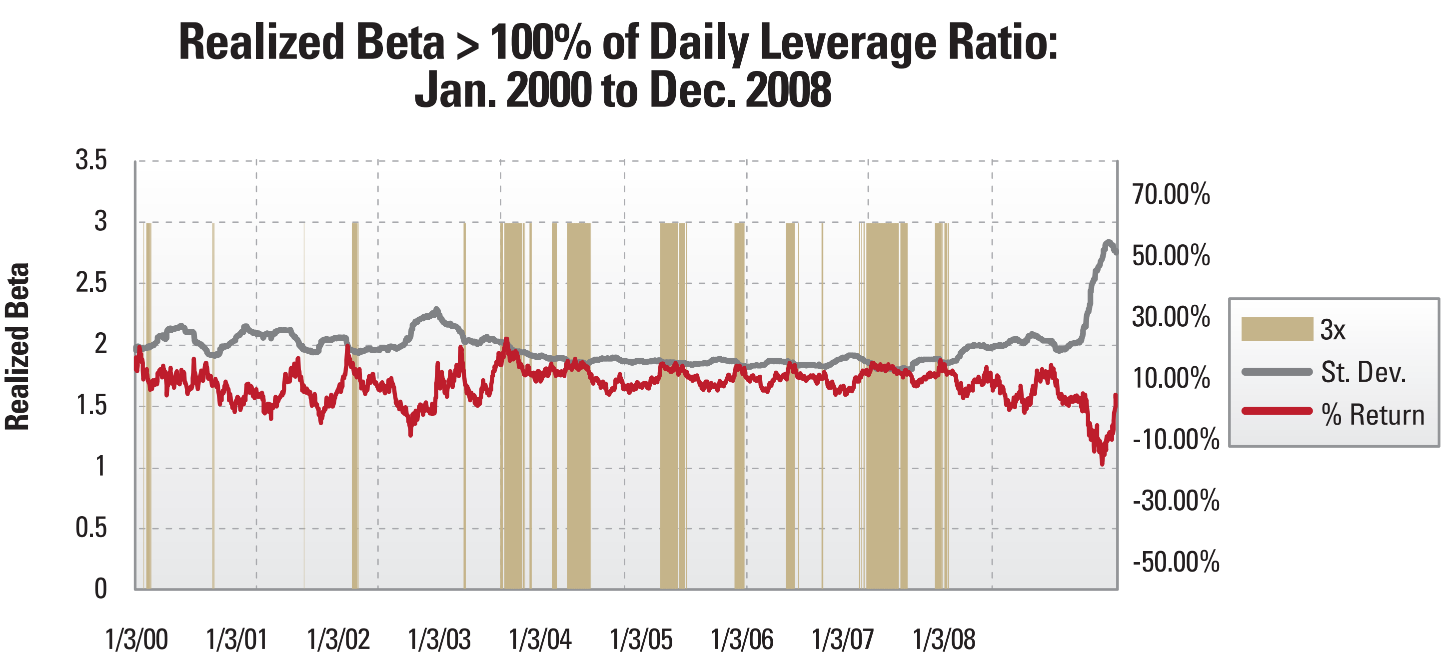


Table 1 details these events more explicitly, including the results for inverse-leveraged funds. There are 2,263 rolling three-month periods from January 2000 through December 2008. To enjoy positive compounding over a three-month time period, the investor needs to not only predict the correct trend, but volatility needs to be constrained as well. For the long investor, there is only a 17% chance of this occurring, while for the short investor, the odds fall to 6%. Thus, the probability of experiencing positive compounding for long-term horizons for leveraged funds is relatively low and investors should not expect to attain a long-term multiple equal to the daily leverage ratio. (For further analysis, including required recovery times, refer to "Required Recovery" in Appendix C.)

Table 1 shows there are 2,263 rolling 3-month periods between January 2000 and December 2008. The number of positive compounding results for 2.0x, 3.0x, -2.0x, and -3.0x levered funds is given.

Total Number of Periods = 2,263	Number of Periods Realized Beta > 100% of leverage (%)
2x	384 (17%)
3x	375 (17%)
-2x	156 (7%)
-3x	135 (6%)

Looking Down the Road

As this study has demonstrated, holding a daily leveraged fund over an extended time horizon can and will often lead to results not commensurate with the daily multiplier. In fact, when volatility becomes extreme, holding periods beyond 10 days become increasingly uncertain, relative to maintaining a particular multiplier. To alleviate this problem, monthly leveraged funds have been and are being brought to the market. These funds will allow investors to achieve a particular multiple on a monthly basis. However, just because a monthly, rather than a daily, multiple is attained, does not mean that these monthly leveraged funds should be held indefinitely.

Figure 8: Probability that realized beta will be within 20% of the leverage ratio of 3.0 for daily and monthly leveraged 3.0x funds, assuming an expected return of 10% and standard deviation of 20%

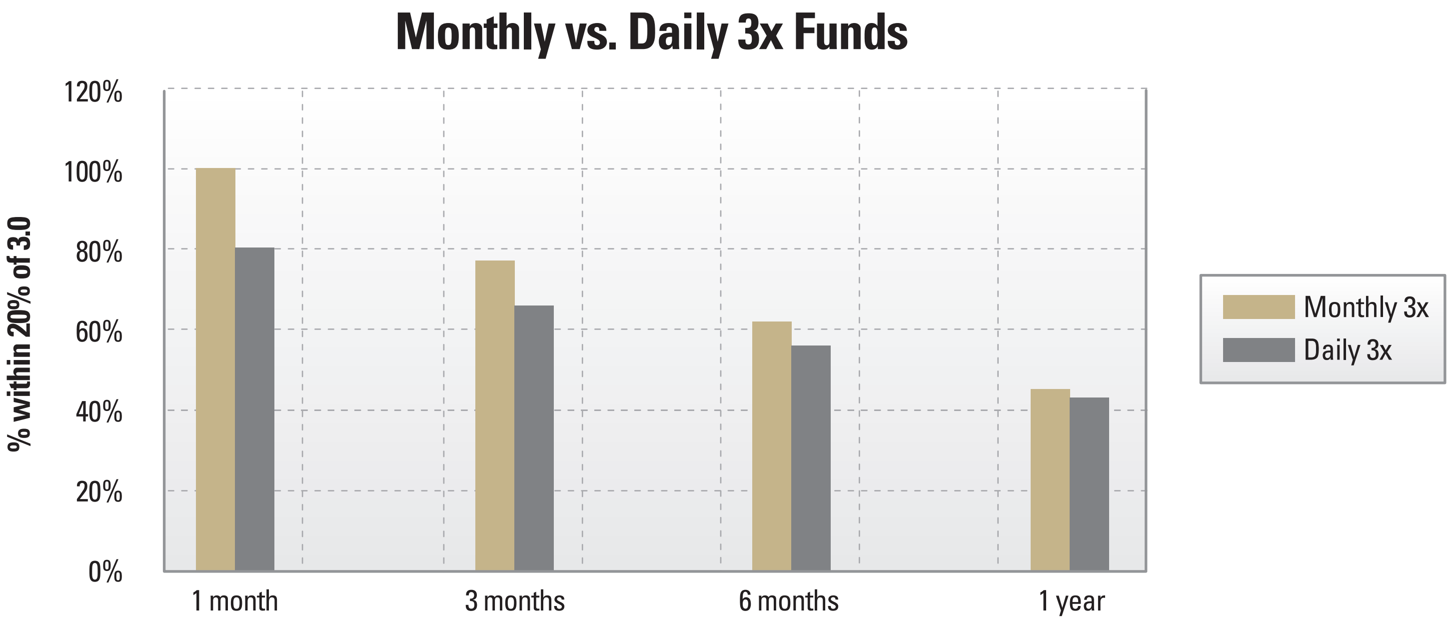


Figure 8 shows the results for a 3.0x daily fund and a 3.0x monthly fund, assuming an expected return of 10% and a standard deviation of 20%. Clearly, a monthly leveraged fund has value for those with extended time horizons. For instance, by 1 month, there is only an 80% chance that a daily leveraged fund will have a realized beta of 3.0, whereas the monthly beta will have a 100% chance that it will be 3.0, ignoring transaction costs and fees. However, the differentiation between the two funds becomes quite blurred when the holding period extends to 1 year.

Although not shown, when the volatility increases to 40%, a 3.0x monthly leveraged fund has a 61% probability of realizing a beta within 20% of 3.0 after 3 months. The daily 3.0x fund only has a 45% chance of this occurring. Relative to the 20% volatility case, the results are 78% for the monthly fund and 66% for its daily counterpart. Put another way, with a three-month holding period, a monthly beta will increase the probability that the investor will be within 20% of the leverage ratio by 16% relative to a daily leveraged fund when the volatility is 20%, and increases the probability by 36% when the volatility is 40%. However, by 1 year, the increase in value of the monthly leveraged fund over the daily leveraged fund again becomes blurred. Similar relational results are found for the -2.0x and -3.0x funds, although the absolute percentages for the funds to be within -2.0 or -3.0, respectively, are much smaller, due to the base and compounding issues. Refer to Figures C10 and C11 in Appendix C for an illustration of this concept.

Thus, for those investors with a time horizon out to 6 months, a monthly leveraged fund clearly has value over a daily leveraged fund. As the time horizon extends to a year or more, regardless of the leverage or volatility, differences between holding a daily or monthly leveraged fund are not dramatically different. The compounding problem becomes increasingly problematic for either fund when dealing with longer time horizons.

Interim Investors

The major issue when investing in a monthly leveraged fund is for those investors buying into the fund between the start date and end date—the latter being when the fund is rebalanced. The monthly leverage is only exact for one particular starting date. For example, if the monthly fund runs from June 1 to June 30, the leverage ratio will only hold for investors starting June 1 and ending June 30, or for those who happen to buy into the fund on a day when the underlying index is at exactly the same value as it was on June 1. Otherwise, the realized leverage ratio will deviate from the underlying fund ratio. For long funds, if the index is above the initial start value, the realized leverage will be less than the underlying fund due to the higher base that one is starting with. If the index is less than the initial start value, the realized leverage will be greater than the underlying index.

For short funds, just the opposite holds. That is, if the index is above the initial start value, the realized leverage will be greater than the underlying fund due to the higher base that one is starting with. If the index is less than the initial start value, the realized leverage will be less than the underlying index.

Table 2: Example of an interim investor within a monthly rebalanced fund not attaining the leverage multiple

	Case 1				Case 2			
	t = 0	t =1	t =2a	t = 2b	t = 0	t = 1	t = 2a	t = 2b
Index	100	102	104	96	100	98	104	96
Index Return		2.00%	1.96%	-5.88%		-2.00%	6.12%	-2.04%
Cumulative Index Return			4.00%	-4.00%			4.00%	-4.00%
3x Return			5.66%	-16.98%			19.15%	-6.38%
3x Beta			2.89	2.89			3.13	3.13
-3x Return			-6.38%	19.15%			-16.98%	5.66%
-3x Beta			-3.26	-3.26			-2.77	-2.77

Table 2 shows both cases for a 3.0x fund and a -3.0x fund. In the first case, the index rises from 100 to 102, then goes to either 104 or 96. In the second case, the index moves to 98, then to either 104 or 96. It is assumed that the investor buys into the monthly fund at time t = 1. Regardless of where the index ends up, the leverage ratio for the 3.0x fund will be 2.89 if bought in at 102. For the -3.0x fund, the leverage will be -3.26. This is due entirely to the base issue. In this case, the base for the interim investor is either 102 or 98. Appendix B mathematically derives what the actual leverage will be and the final equation is:

$$\text{Beta} = (L-1)/(1+Lr_t) + 1$$

where L is the leverage ratio and r_t is the return of the underlying index from when the fund is rebalanced to when the fund is actually purchased within the rebalancing period.

There has been some confusion about how much leverage an interim investor may experience. Bush[4] incorrectly compares the interim-leveraged ETF return to the underlying index return over the total period, suggesting leverage could be as much as 10.0x for a 2.0x fund, or even negative. However, it is theoretically impossible when comparing the ETF return from the period the investor bought into the market with the corresponding index return from that point forward, to have a beta of the opposite sign for an interim investor. The only case where this could occur is if it is assumed the market falls 50% (33.34% for a 3.0x fund), and theoretically a 2.0x (3.0x) fund is then completely wiped out anyway. Looking at an extreme, but more likely, example, such as assuming at t = 1 the index value is 125, a monthly 2.0x fund will have a beta of 1.67, and a 3.0x fund will have a leverage of 2.14. Now it is certainly possible that leverage can be much higher than expected. For the short fund in this case, the leverage is -5.0 for the -2.0x fund and -15.0 for the -3.0x fund.

The main advantage of buying into the monthly leveraged fund within the month over a daily leveraged fund is that only the base problem has to be dealt with, assuming your investing time horizon is to the end of the month. The daily compounding problem is completely alleviated for the time remaining until the monthly fund resets. However, one should calculate the leverage they are buying into at the time of the investment. It could be much higher or lower than expected.

It also should be noted that the example above assumes the monthly leveraged fund will be priced perfectly each day: i.e., maintaining a constant leverage ratio throughout the month. This may not be the case, since the leverage is specified to be a particular levered amount of the index return on a particular date, not in the interim between the dates. Because of this, investor expectations could factor into the price of the monthly levered beta, as it does not specify that it will be equal to the levered value of the index return, except on one particular date.

Putting It All Together

Based on the three theorems developed in this paper, a few broad conclusions can be made:

1. With all else being equal, a trending market is beneficial to investors in leveraged funds who make correct market predictions. This holds for both short-term and longer-term investors in these funds, assuming volatility is relatively constrained.
2. The greater the volatility, the greater a long-term holding period's realized beta will diverge from the daily leverage ratio.
3. In highly volatile markets, both leverage level and time frame should be considered carefully. Investors with horizons longer than 10 to 30 days should be especially cautious. For investors concerned with performance relative to the underlying benchmark, a time frame of 30 days or less should be considered for a 2.0x fund. Investors using 3.0x leveraged funds cannot assume realized betas will maintain their corresponding daily leverage ratios to any degree of certainty beyond 10 days.
4. Under normal market conditions with an average volatility of 20%, time frames out to three months or more are possible, without any serious degradation in realized beta values. However, investing in levered funds based only on a market projection is insufficient. Investors must also consider the path of the market to the ultimate projected value. Specifically, projecting volatility is also critical. With high volatility, time frames of less than 30 days are advised. Under "normal" market conditions (volatility of 20% or less), longer time frames combined with higher leverage can be considered.
5. Although monthly rebalancing does extend the time frame an investor can consider holding leveraged ETFs, these funds have results very similar to daily rebalanced levered funds after six months. Investors buying into these funds during the month should not expect the initial leverage ratio to hold until the end of the month. The actual leverage the investor is buying is based on the value of the underlying index relative to the value of the index at the beginning of the month. This ratio may be higher or lower and can be substantially different than the fund's leverage ratio, depending on the value of the underlying index relative to the value when initially rebalanced.

[4] See Bush in Appendix D for more information.

Appendix A: Proving Theorems 1-3

Theorem 1: The greater the trend, the greater the beta divergence. For both bullish and bearish levered funds, a positive trend will result in positive beta divergence, and a negative trend will result in negative beta divergence. These divergences will increase over time.

Let r_i be the return each day, v be the daily leverage, and N the number of days. The return of any underlying index (ID) and any ETF over time N, assuming no volatility, will be:

$ID_N/ID_0 - 1 = (1+r)^N - 1$

$ETF_N/ETF_0 - 1 = (1+vr)^N - 1$

Beta divergence (BD) for any levered fund over any time frame is defined as $(ETF_N/ETF_0 - 1) / (ID_N/ID_0 - 1) - v$. This is equal to $[(1+vr_i)^N - 1]/[(1+r_i)^N - 1] - v$. Assuming no volatility, then r_i is constant and

$BD = [(1+vr)^N - 1]/[(1+r)^N - 1] - v$. When N = 1, BD = 0 and there is no beta divergence.

For the two-period case, $BD = [(1+vr)^2 - 1]/[(1+r)^2 - 1] - v = [vr(2+vr)]/[r(2+r)] - v = v(2+vr)/(2+r) - v$.

Thus, if r is positive and increases, BD must increase, assuming positive or inverse leverage ($v > 1.0$ or $v < 0.0$), since $[v(2+vr)/(2+r) - v] > 0$ for all values of positive r. If r is negative, BD must be negative, since $[v(2+vr)/(2+r) - v] < 0$ for all values of negative r and $v > 1.0$ or $v < 0.0$. The more negative the r, the more negative BD will be. In addition, the greater the trend, the greater is BD in absolute value. A positive trend will result in positive beta divergence, and a negative trend will result in negative beta divergence.

To generalize for any time period n, note that the value of

$(1+vr)^N - 1 \approx vNr - v^2Nr^2/2 + v^2(Nr)^2/2 = vNr + v^2 r^2 N(N-1)/2$ and

$(1+r)^N - 1 \approx Nr - Nr^2/2 + (Nr)^2/2 = Nr + r^2N(N-1)/2$. Dividing the two and subtracting v begets BD, which is

$BD = [vNr + v^2 r^2 N(N-1)/2]/[Nr + r^2N(N-1)/2] - v$

Thus, if r is positive, BD must be positive, since $[vNr + v^2 r^2 N(N-1)/2]/[Nr + r^2N(N-1)/2] - v > 0$ for $v > 1.0$ or $v < 0.0$.
If r is negative, BD must be negative, since $[vNr + v^2 r^2 N(N-1)/2]/[Nr + r^2N(N-1)/2] - v < 0.0$ for $v > 1.0$ or $v < 0.0$.
In addition, the greater the trend, the greater BD will be in absolute value. Q.E.D.

Theorem 2: **The greater the leverage, the greater the trend effect.**

Since BD = $[vNr + v^2 r^2 N(N-1)/2]/[Nr + r^2N(N-1)/2] – v$, the first term $[vNr + v^2 r^2 N(N-1)/2]/[Nr + r^2N(N-1)/2] > v$ when r and v are positive ($v > 1.0$). Thus, BD must increase as v increases. When v is negative and r is positive, the first term is always positive for $v < -1.0$, and smaller than v when between 0 and -1.0; thus, as v increases (becomes more negative), BD increases. When r is negative and v is positive, the first term is always smaller than v, so BD is negative and becomes increasingly so as v becomes more negative. Finally, if both v and r are negative, the first term is always less than v–i.e., more negative–so BD is negative and becomes increasingly so as v becomes more negative. Thus, the greater the leverage, the greater the trend effect, as measured by the absolute value of BD. Q.E.D.

Theorem 3: **The greater the volatility of the underlying index, the greater the decline in median beta values relative to their daily leverage ratios. In addition, the greater the leverage and the longer the time frame, the greater volatility's effect will be.**

Mathematically, this states that beta divergence will decrease for long funds and increase for short funds. For example, a 3x volatility increases and the realized beta declines from 2.8 to 2.6. Beta divergence as defined decreases from 2.8 – 3.0 = -0.2 to 2.6 – 3.0 to -0.4. Alternatively, assume a -3.0x short fund's realized beta changes from -2.8 to -2.6. Beta divergences increase from -2.8 – -3.0 = 0.2 to -2.6 – -3.0 = 0.4.

To prove the above theorem, it must be noted that compounded returns are lognormally distributed. To calculate future expected returns and median values, discrete returns must be converted into their continuous counterparts. Continuous confidence intervals can then be computed, and these can then be converted back into discrete values. To calculate median values, the confidence interval is set to zero. Converting expected returns and standard deviation into their continuous counterparts begets the following:

For the underlying index:

(1) $\mu_{cx} = \ln(1 + \mu_d) - \sigma^2_{cx} /2$

(2) $\sigma_{cx} = \sqrt{\ln[\sigma^2_d/(1+\mu_d)^2 + 1]}$

For the leveraged fund:

(3) $\mu_{cv} = \ln(1 + v\mu_d) - \sigma^2_{cv} /2$

(4) $\sigma_{cv} = \sqrt{\ln[v\sigma^2_d/(1+v\mu_d)^2 + 1]}$

where μ_{cx} and μ_{cv} = continuous expected return for the index and levered fund, respectively

μ_d = discrete expected return

σ_{cx} and σ_{cv} = continuous standard deviation for the index and levered fund, respectively

σ^2_d = discrete variance

Note that the continuous mean is a function of the variance. This is because the lognormal distribution is not symmetric and the value of the variance affects the spread of the distribution, which in turn affects the mean. Once the above values are found, the standard deviation associated with any time frame can be found by applying the following:

(5) $\sigma_{tax} = \sigma_{cx}/N^{.5}$ and $\sigma_{tav} = \sigma_{cv}/N^{.5}$

where σ_{tax} and σ_{tav} = the time-adjusted standard deviation for the index and levered fund, respectively, while N = the time into the future for which the confidence interval or probability estimate is being calculated. To find the range of continuous values(C), just add and subtract the number of standard deviations associated with a particular confidence level:

(6) $C_x = \mu_{cx}$ +/- $Z(\sigma_{tax})$ and $C_v = \mu_{cv}$ +/- $Z(\sigma_{tav})$

where μ_{cx} and μ_{cv} are the continuous expected returns as defined in equation (1), C_x and C_v are the continuous values for the index and levered fund, respectively, and Z is found from the standard normal distribution table. For median values, Z is equal to zero. This can also be generalized even further to show that the confidence intervals will also increase with any increase in volatility.

To find the discrete confidence interval ranges for the index and levered fund, simply convert the continuous values back into discrete values D_x and D_v by using the inverse natural logarithm function, where e = 2.71828 and raise to the n^{th} power:

(7) $D_x = (e^{Cx})^N$ and $D_v = (e^{Cv})^N$

where D_x and D_v are the discrete values for the index and the levered fund, respectively.

Beta divergence can now be calculated as:

(8) $BD = D_v/D_x - v = (e^{Cv})^N/(e^{Cx})^N - v$.

To find the median values, C_x and C_v will equal μ_{cx} and μ_{cv}, respectively, so
$BD = (e^{\mu cv})^N/(e^{\mu cx})^N - v$. Plugging in equations (1) and (3) for $e^{\mu cx}$ and $e^{\mu cv}$ begets:

(9) $BD = [\exp(\ln(1 + v\mu d) - \sigma^2_{cv} /2)^N]/[\exp(\ln(1 + \mu_d) - \sigma^2_{cx} /2^N]$

where exp() is $e^{()}$. Now plugging equations (2) and (4) for σ^2_{cx} and σ^2_{cv} begets:

$$(10)\ BD = \frac{[\exp(\ln(1 + v\mu d) - \{\sqrt{\ln[v\sigma^2_d/(1+v\mu_d)^2 + 1]}\}^2 /2)^N] - v}{[\exp(\ln(1 + \mu_d) - \{\sqrt{\ln[\sigma^2_d/(1+\mu_d)^2 + 1]}\}^2 /2^N]}$$

Now although this is a fairly complicated term, BD is now a function of only v, μ_d, and σ^2_d. In both the numerator and denominator of equation (10), σ^2_d reduces their respective values. Since $v\sigma^2_d$ in the numerator is greater than σ^2_d in the denominator, the numerator falls by more than the denominator, as σ^2_d increases because they are subtracted from exp(.). Thus, the greater the volatility of the underlying index, the smaller the median beta divergence will be for a long-levered fund, since v will be positive, and the greater the median beta divergence will be for a short fund, since v will be negative. Q.E.D.

Appendix B: Intra-Month Leverage Calculations

Mathematically, given any initial index value and the value of the index when one buys into the monthly fund, the leverage realized from that date forward to the end of the month can be calculated as follows:

Let $X_n = X_0(1 + r_n)$ and also $X_n = X_t(1 + r_2)$, and $X_t = X_0(1+r_1)$

where the subscript 0 is for the beginning of the month, time t can be anytime within the month, and N is at the end of the month. X is the underlying index and r is for the return. There will be an r_1 , r_2, and an r_N to designate the return from t = 0 to t =1, from t =1 to t = N, and from t = 0 to t = N.

Letting L stand for the monthly leverage, then a levered fund value at time t and N—assuming no deviations from a continuous leverage value equal to the funds' leverage throughout the month—will be:

$LX_t = [L (X_t/X_0 -1) + 1]^* X_0$

$LX_N = [L (X_n/X_0 -1) + 1]^* X_0$

A levered fund's return from time t to N will be:

$LR = LX_N/LX_t -1$

A levered fund's interim beta or implied leverage for investors not buying at time = 0 will be:

$Beta = LR/r_2$

Substituting terms, one finds that the realized beta for an interim investor will be strictly dependent on the value of the index at time t relative to time 0 and can be rewritten as:

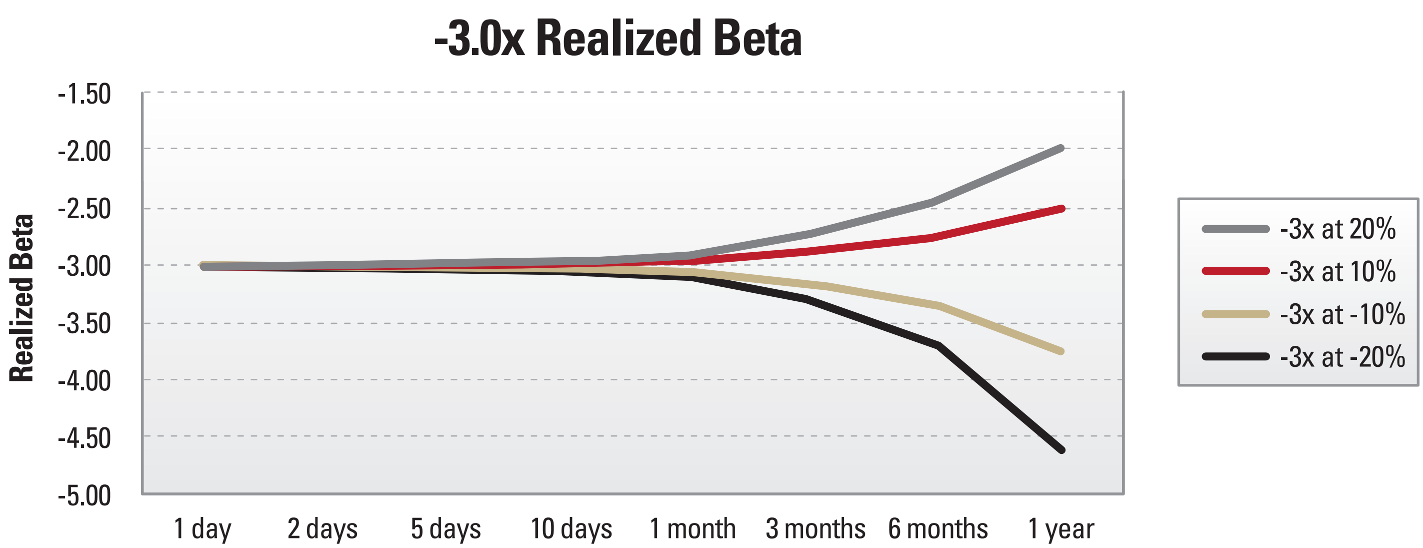
$Beta = (L-1)/(1+Lr_t) + 1$

Appendix C: Ancillary Material

Theorem 1: Negative Beta Divergence When Market Trends Against the Investor

A -3.0x fund will have a loss less than -3.0x the underlying index when the index trends up, and will have more than a 3.0x gain when the index trends down. Figure C1 shows nominal annual trends from -20% to 20% for a -3.0x fund. As an example, when the market is trending down in a non-stochastic manner at -20%, the realized beta for a -3.0x fund is -3.68 after six months, resulting in a negative beta divergence (-3.68 – -3.0 = -0.68). This occurs since the market is trending down. However, in this case, since the fund is short, the negative beta divergence is a positive for the investor, as more than -3.0x the cumulative return of the index is realized. Thus, all else being equal, and in conditions of very low or zero volatility, a trending market is a positive for the multi-day investor. Gains are magnified for a correct market call and cumulative losses are limited relative to the daily leverage ratio.

Figure C1: Realized -3.0x beta over time, assuming a non-stochastic -20% to +20% nominal annual average trend



Theorem 2: The Effect of Trends – -1.0x vs. -3.0x Funds

Figure C2, on the next page, shows the realized beta value for a -1.0x fund and a -3.0x fund. If the market trends down, the -3.0x realized beta after 6 months is -3.68. Alternatively, three times the -1.0x beta is only -3.32. Thus, assuming a negative non-stochastic trend, one-third of the investment in a -3.0x fund will outperform three times the investment in a -1.0x fund. Alternatively, the losses will be less if the market trends up.

Figure C2: Realized beta value over time, assuming a 20% and -20% nominal annual non-stochastic trend. The -1.0x beta is multiplied by 3 to demonstrate that a 3.0x fund's beta will increase relative to a -1.0x fund's beta.

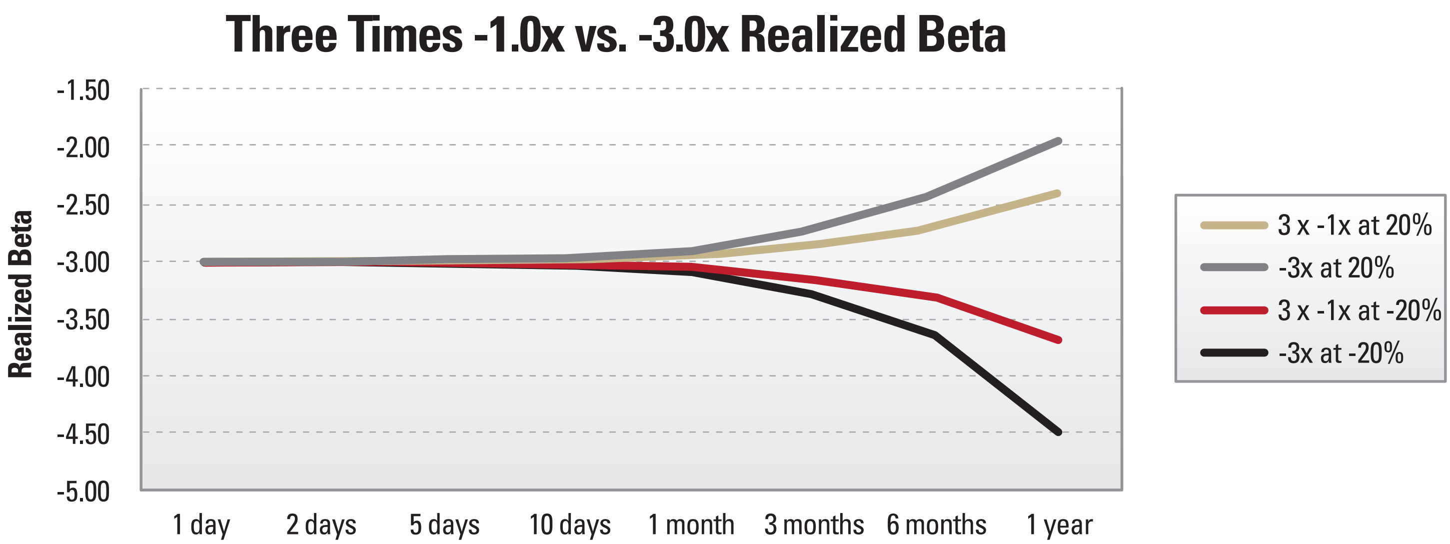
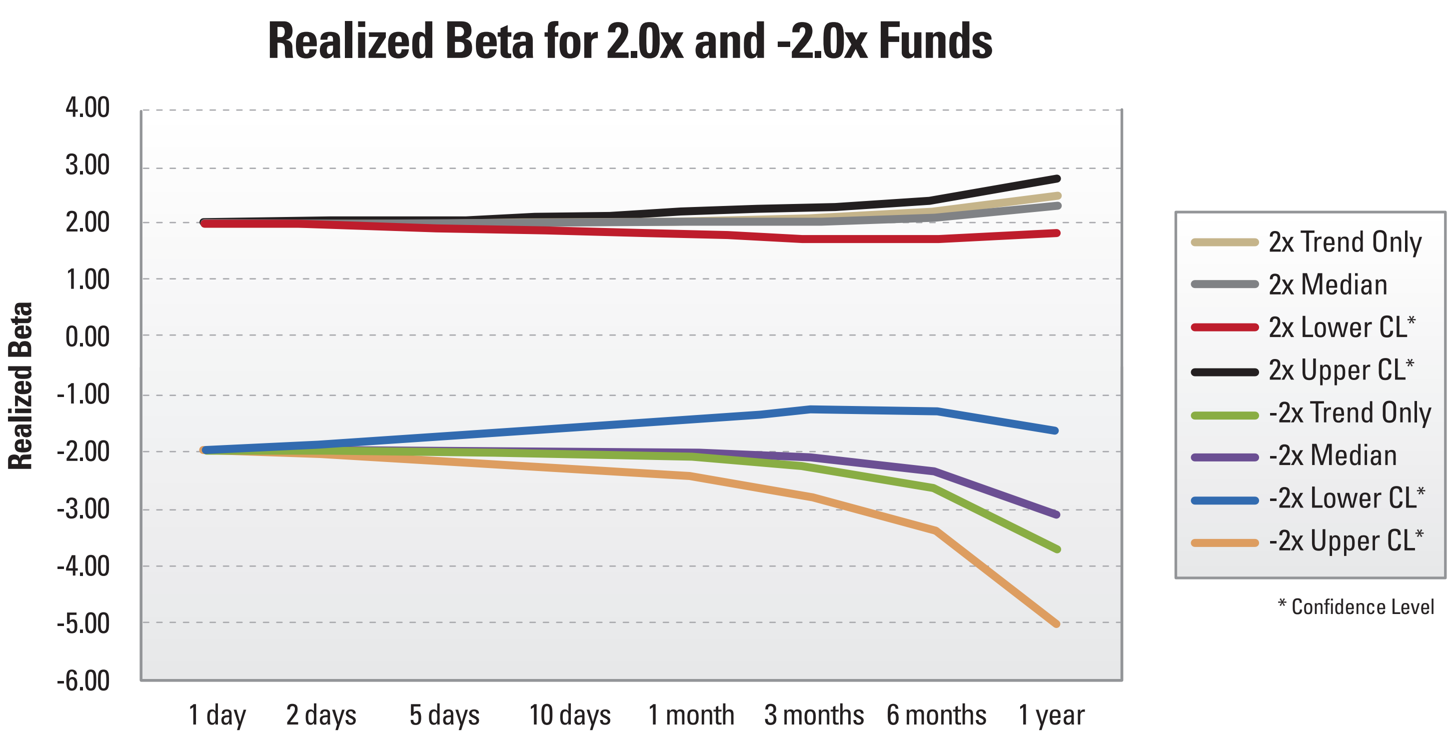


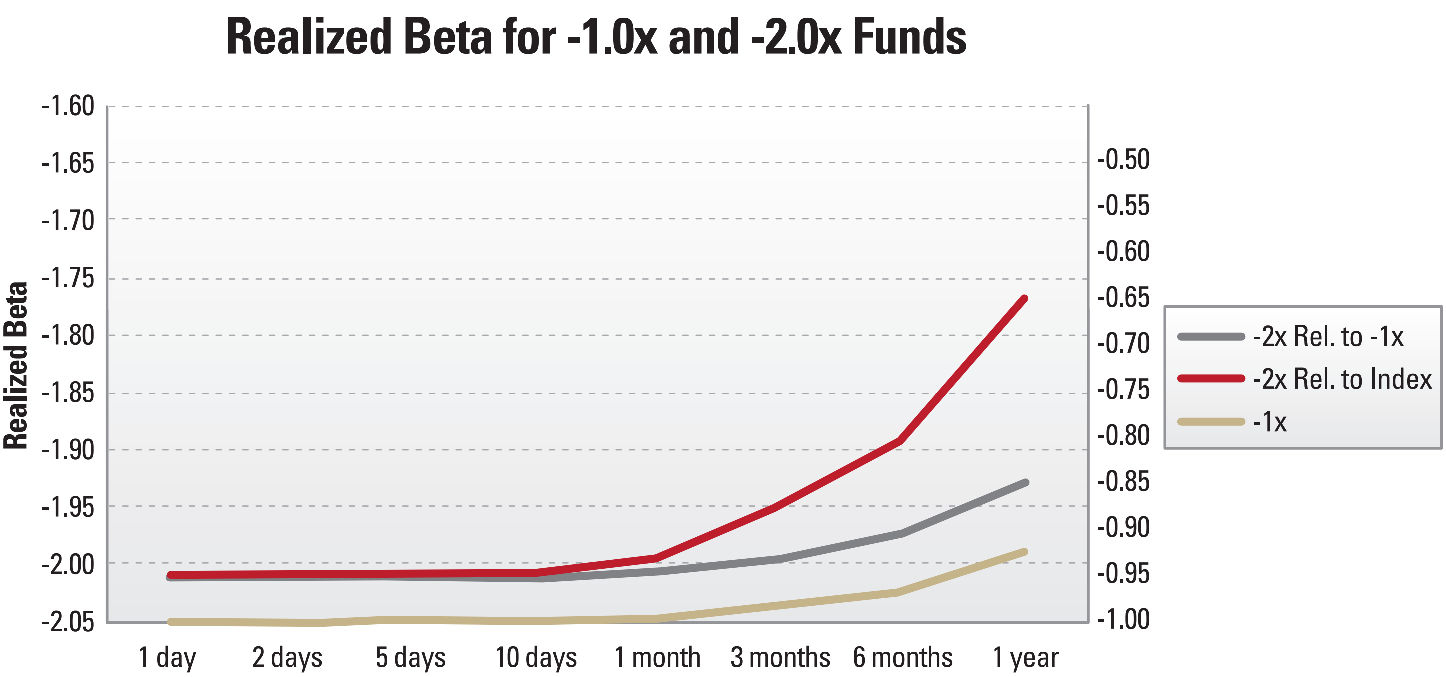
Figure C3: Realized beta for 2.0x funds, assuming a 40% trend; and -2.0x funds, assuming a -40% trend, along with 20% volatility



Theorem 3: Beta Divergence

Figure C4 visually details the beta divergence for a -2.0x fund over one year, assuming a 10% return and 20% volatility. As one can see, the realized beta for a -1.0x fund does not maintain a -1.0 ratio. By 1 year, the median value has fallen to -0.93. For the -2.0x fund, the difference between the -1.0x fund's realized beta and the value of -1.0 is the base problem, and the difference between the -2.0x fund and the -1.0x fund is the standard leverage compounding problem. To graphically demonstrate this, the realized beta for the -2.0x fund is computed relative to the -1.0x fund and to the underlying index. The added beta divergence relative to the index is the base issue, whereas the divergence relative to the -1.0x fund is the leverage component of the compounding problem. Long-leveraged funds do not have this base issue.

Figure C4: Realized beta for -1.0x and -2.0x funds, assuming a 20% trend and 20% volatility. Right-hand axis is for the -1.0x fund.



Historical Periods & Their Effects on Leveraged Funds

To compare what would have happened to a leveraged fund based on historical data, four unique periods are taken from the above time period. They are:

1. Jan. 12, 2000 to April 12, 2000, which had a normal trend (11.6%) and normal volatility (23%).
2. Aug. 26, 2002 to Nov. 21, 2002, which had a normal trend (8%) with high volatility (30%).
3. Aug. 6, 2003 to Nov. 3, 2003, which had high trend (34%) with low volatility (12%).
4. Sept. 30, 2008 to Dec. 29, 2008, which had high trend (-68%) with high volatility (68%).

Although different results can occur under each of these conditions because the sequence of returns matters, the following results are indicative of what an investor would have experienced in the past and may experience in the future. Figures C5 through C8 show the theoretical realized betas for -3.0x, -2.0x, 2.0x, and 3.0x levered funds for the above-referenced three-month time periods. The cumulative returns are shown in percentages.

Figure C5 shows that, even with normal trend and normal volatility, realized beta can vary dramatically, even within a three-month period. However, by 3 months, the only levered fund that is more than 20% different from its daily leverage ratio is the 3.0x fund. The only extreme values in realized beta occur when the cumulative return is close to zero, so the economic significance of those events is minimal.

Figure C5: Normal trend (11.6%) with normal volatility (23%)

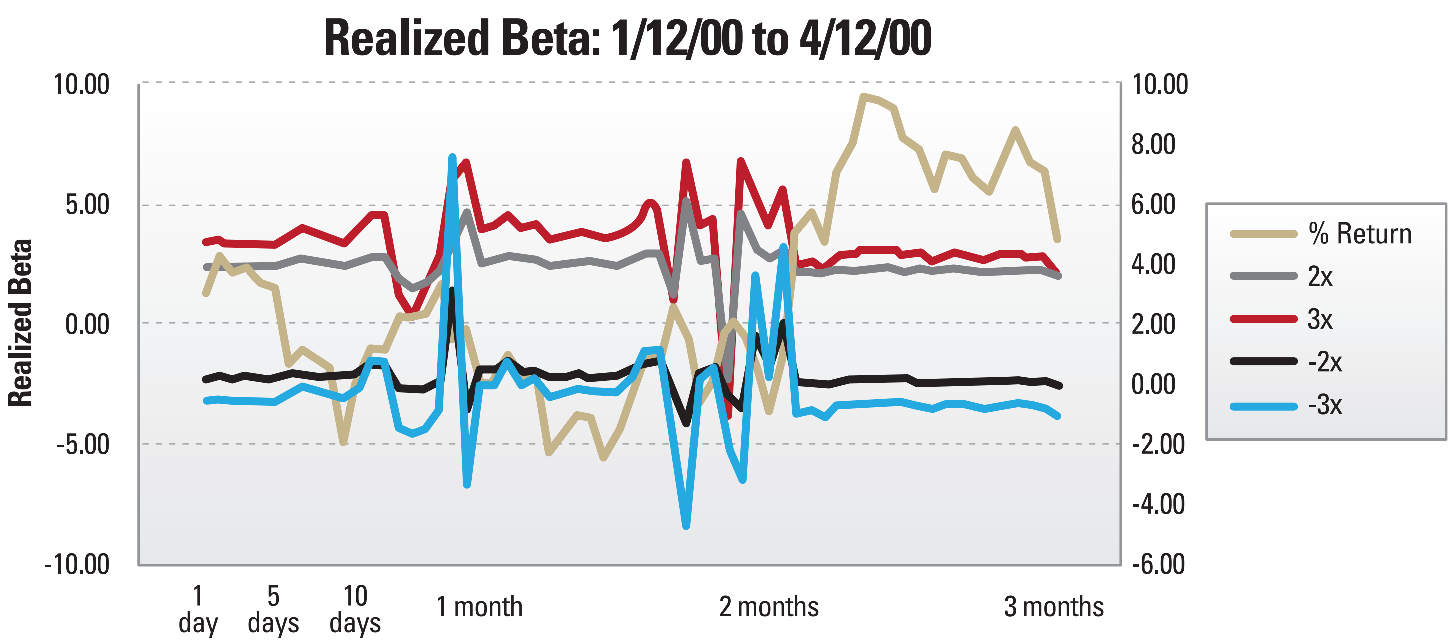


Figure C6 on the following page demonstrates qualitatively the same results as Figure C5, suggesting that even with slightly less trend and higher volatility, realized beta can be relatively stable. However, as the cumulative returns start bouncing around zero, the realized beta values became quite volatile and, on any given day, the realized beta diverges a great deal from the daily leverage ratio for all of the funds.

Figure C6: Normal trend (8%) with high volatility (30%)

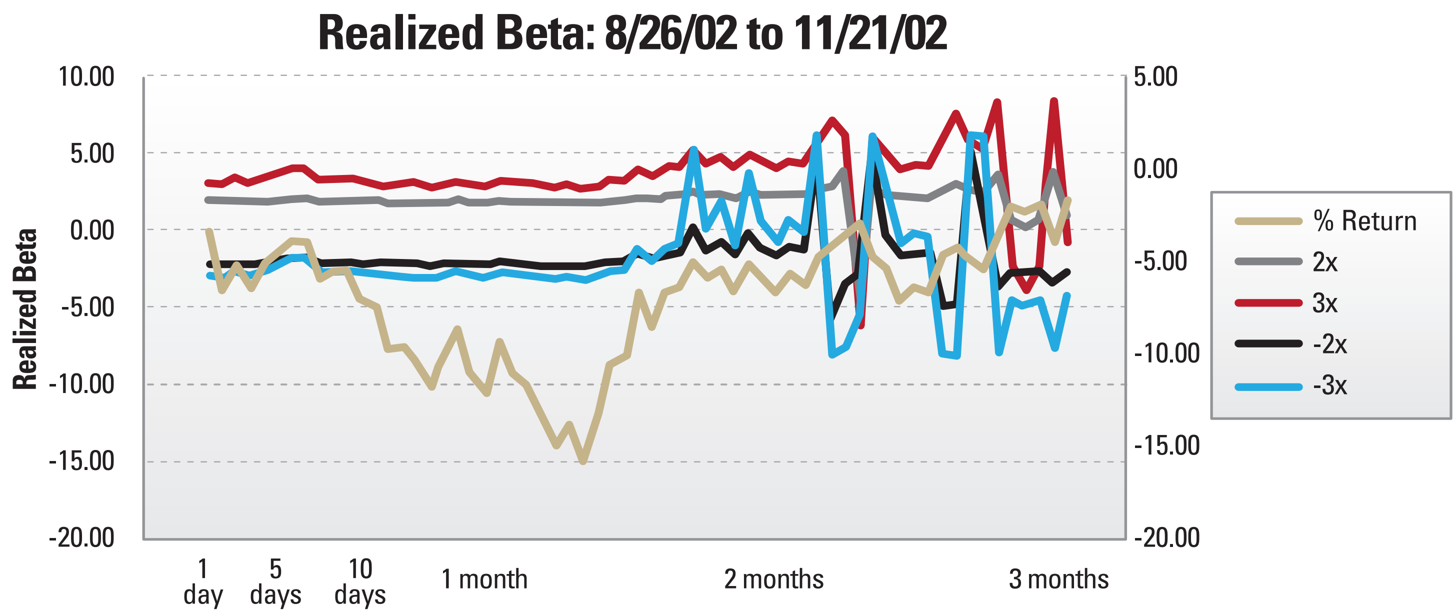


Figure C7 shows the optimal conditions for longer-term investors in levered funds. With high trend and low volatility, all the levered funds' realized betas diverge very little from their respective daily leverage ratios, even after three months.

Figure C7: High trend (34%) with low volatility (12%)

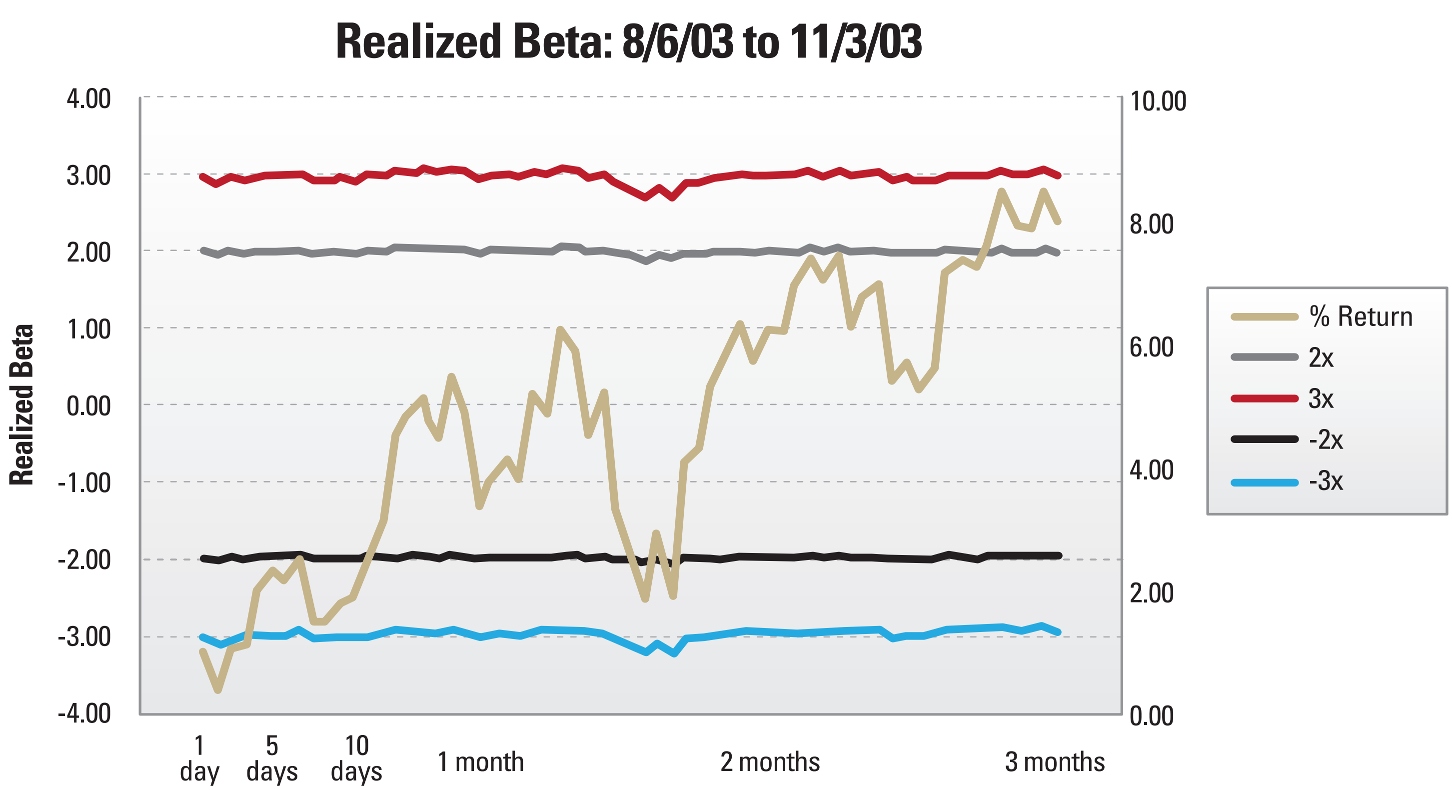
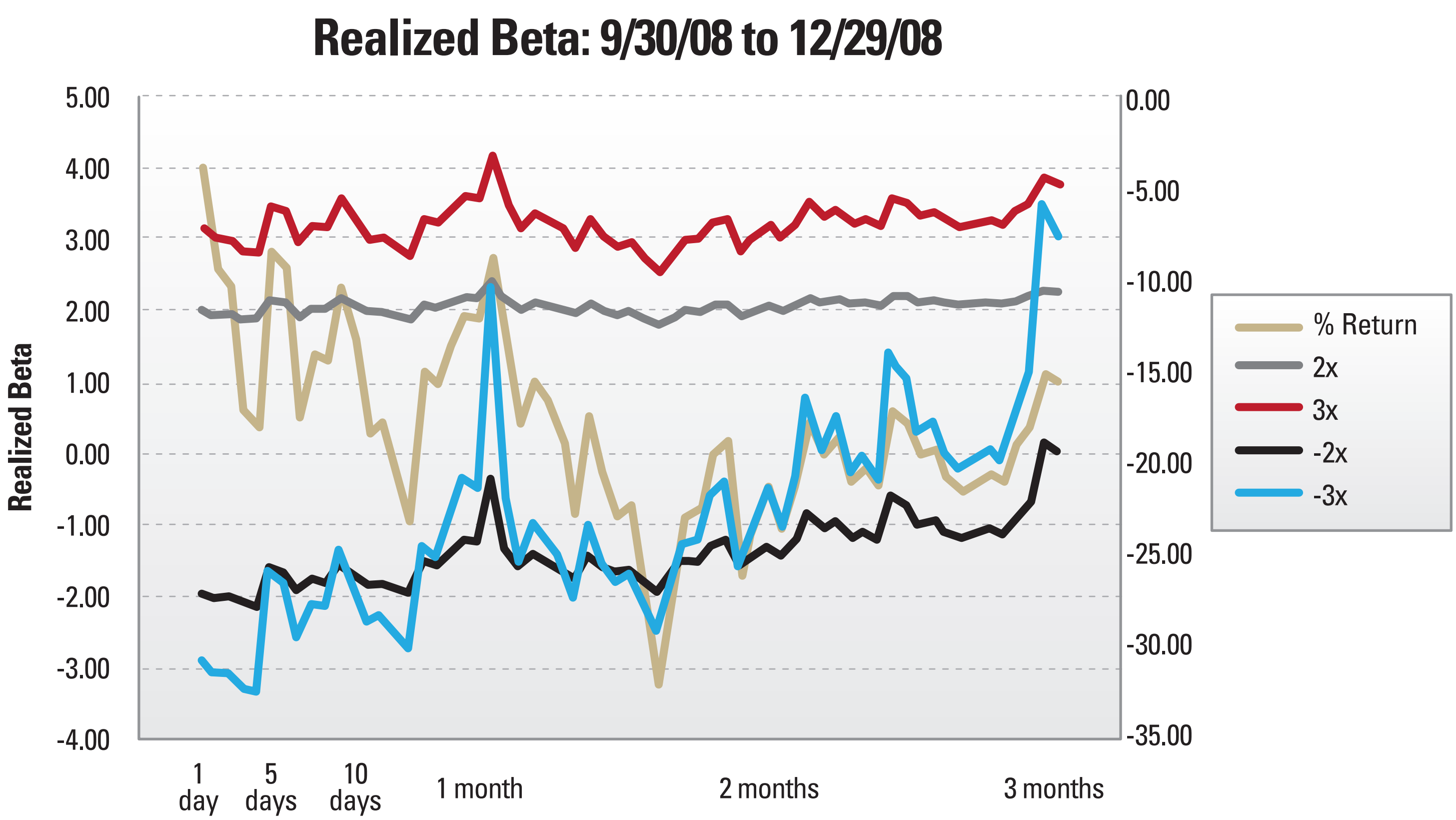


Figure C8 also shows the optimal trend conditions, but the worst volatility conditions. The 2.0x and 3.0x funds both stay within a 20% divergence, but both magnify the cumulative loss more than the daily leverage ratio, 2.3x and 3.7x, respectively. What is so dramatic is that by 1 month, the -3.0x fund ends up losing money during a short-term market recovery, even though the index is still down cumulatively by almost 10%. By 3 months, both the -2.0x and -3.0x short-leveraged funds have lost money. This is the result of a continual degradation of their respective realized betas, which becomes increasingly severe after 10 days. The decline of these funds' realized betas to well below the daily leverage ratio suggests, once again, that with high volatility, regardless of the accuracy of the trend prediction, time frames past 10 days are ill advised.

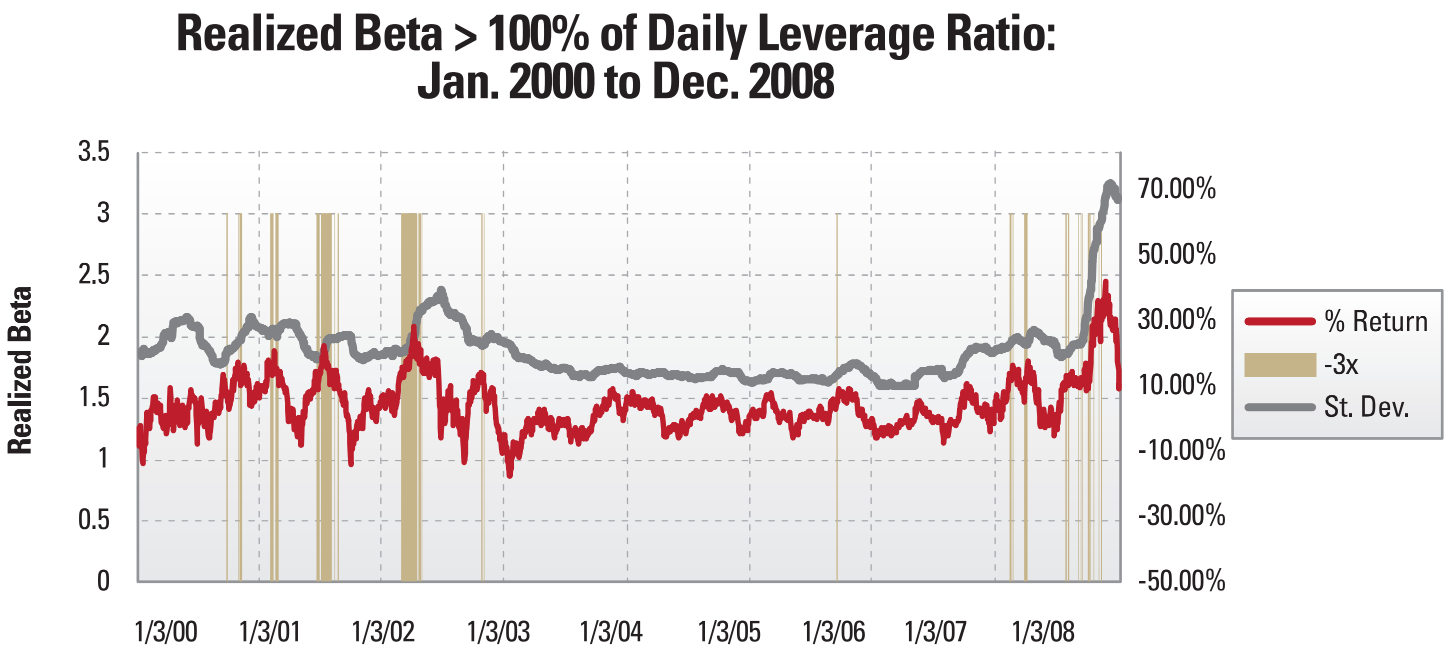
Figure C8: High trend (-68%) with high volatility (68%)



-3.0x Fund Trends: Jan. 2000-Dec. 2008

Figure C9 on the next page shows that, in almost all instances, a trend of -10% for 3 months is needed and from early 2003 to late 2008, there are virtually no periods where positive compounding is realized. Interestingly enough, despite the increase in volatility in late 2008, the trend is so negative that there are a number of positive compounding periods during this time.

Figure C9: Three-month periods in which realized beta is greater than 100% of daily leverage ratio, given the market trend is down for a -3.0x fund. Time periods are delineated by columns equal to 3.0 when this occurs along with inverse market returns (i.e., -10% is delineated as 10%).



Required Recovery

To further delineate the difficulty of attaining, maintaining, or recovering to a realized beta equal to the daily leverage ratio over any length of time, the probability of a realized beta falling to less than 80% of the daily leverage ratio anytime over a three-month period is calculated for 2.0x, 3.0x, -2.0x, and -3.0x funds. Over the next three-month period, the probability of the realized beta recovering to the daily leverage ratio at any time within that period is also calculated, assuming one of two underlying economic conditions holds:

1. Normal expected trend with normal volatility; or
2. High trend with normal volatility to demonstrate that favorable conditions are no guarantee of success.

Probabilities for either event are based only on when the market moves in a direction favorable to the fund, since the investor would prefer the realized beta in absolute value to be close to or exceed the daily leverage ratio. To remove economically insignificant events, the realized return must be greater than 1%. Ten thousand (10,000) Monte Carlo simulations are used to estimate the probabilities.

Table 3 on the following page shows the probability that the market is up at 3 months is 58%, given 10%/20% trend/volatility conditions. During these three months, there is a 40% chance that sometime within this period, the realized beta for a 2.0x fund will be 80% or less than the daily leverage ratio. If this probability is realized by the investor, there is only a 23% chance that, over the next three months from the date of occurrence, the leverage fund's realized beta will recover to the daily leverage ratio number. For the 3.0x fund, the probability of falling to 80% of the daily leverage ratio is 62%, with only a 30% chance of recovery. The numbers are relatively similar on the short side, but with higher probabilities of falling below 80%. This is the result of the base issue discussed in the volatility section earlier.

Under higher volatility, the probability of falling to less than 80% increases, although recovery probabilities are approximately the same. For the 3.0x fund, the probability of falling to less than 80% increases from 62% to 78%. Even under more optimal conditions, such as a 20% trend, the probability that, sometime within a three-month holding period, a 3.0x fund will have a realized beta of less than 80% of the daily leverage ratio is still around 60%. The chance of this recovering does increase, but only from 30% to 38%. Thus, investors who realize a significant decline in the realized beta relative to the daily leverage ratio should not expect, with any high probability, that the realized beta will recover to its daily leverage ratio over time.

Table 3: Probabilities of realized beta falling to 80% of leverage ratios and recovery outcomes

Realized Beta: 10% trend, 20% volatility	Probability market is up for long and down for short by more than 1%	Probability realized beta less than 80%, given correct market call	Probability realized beta recovers to 100%
2.0x	58%	40%	23%
3.0x	58%	62%	30%
-2.0x	42%	64%	24%
-3.0x	42%	82%	26%

Realized Beta: 10% trend, 40% volatility	Probability market is up for long and down for short by more than 1%	Probability realized beta less than 80%, given correct market call	Probability realized beta recovers to 100%
2.0x	50%	69%	25%
3.0x	50%	78%	30%
-2.0x	50%	81%	33%
-3.0x	50%	87%	34%

Realized Beta: 20% trend, 20% volatility	Probability market is up for long and down for short by more than 1%	Probability realized beta less than 80%, given correct market call	Probability realized beta recovers to 100%
2.0x	67%	39%	28%
3.0x	67%	61%	38%
-2.0x	33%	60%	18%
-3.0x	33%	79%	20%

Figure C10: Probability that realized beta will be within 20% of the leverage ratio of -2.0 for daily and monthly leveraged -2.0x funds, assuming an expected return of 10% and standard deviation of 20%

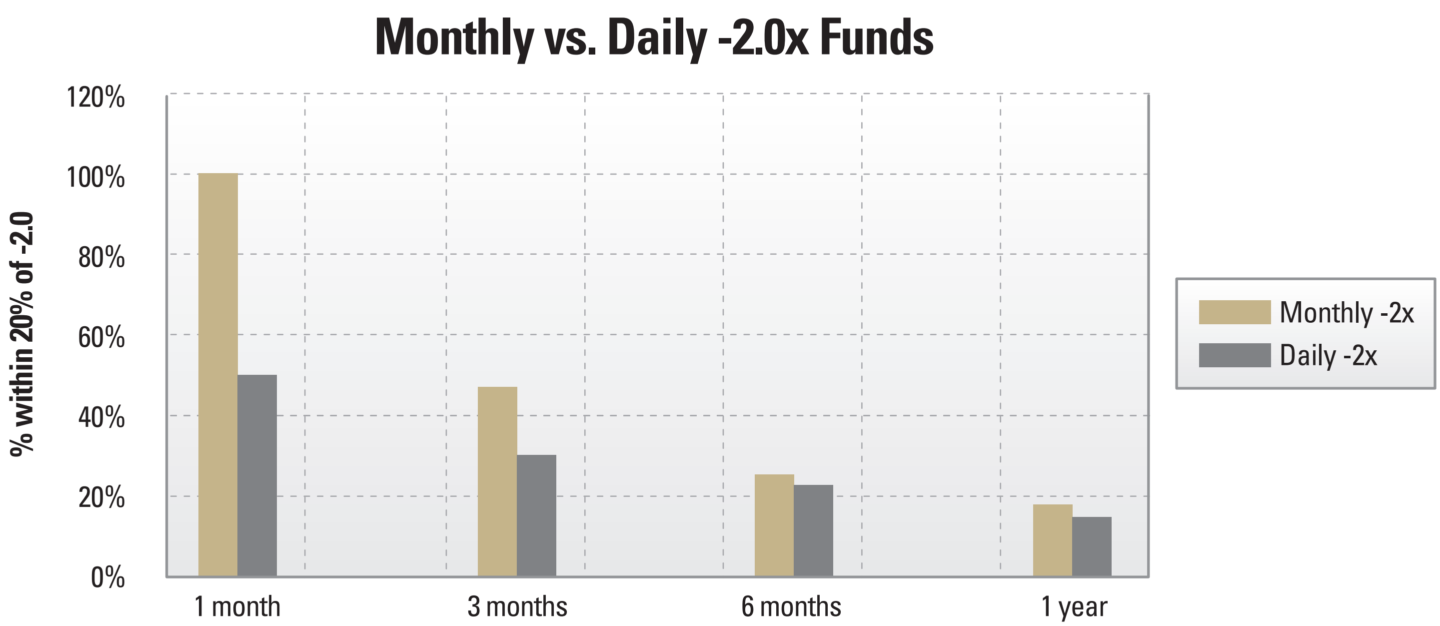
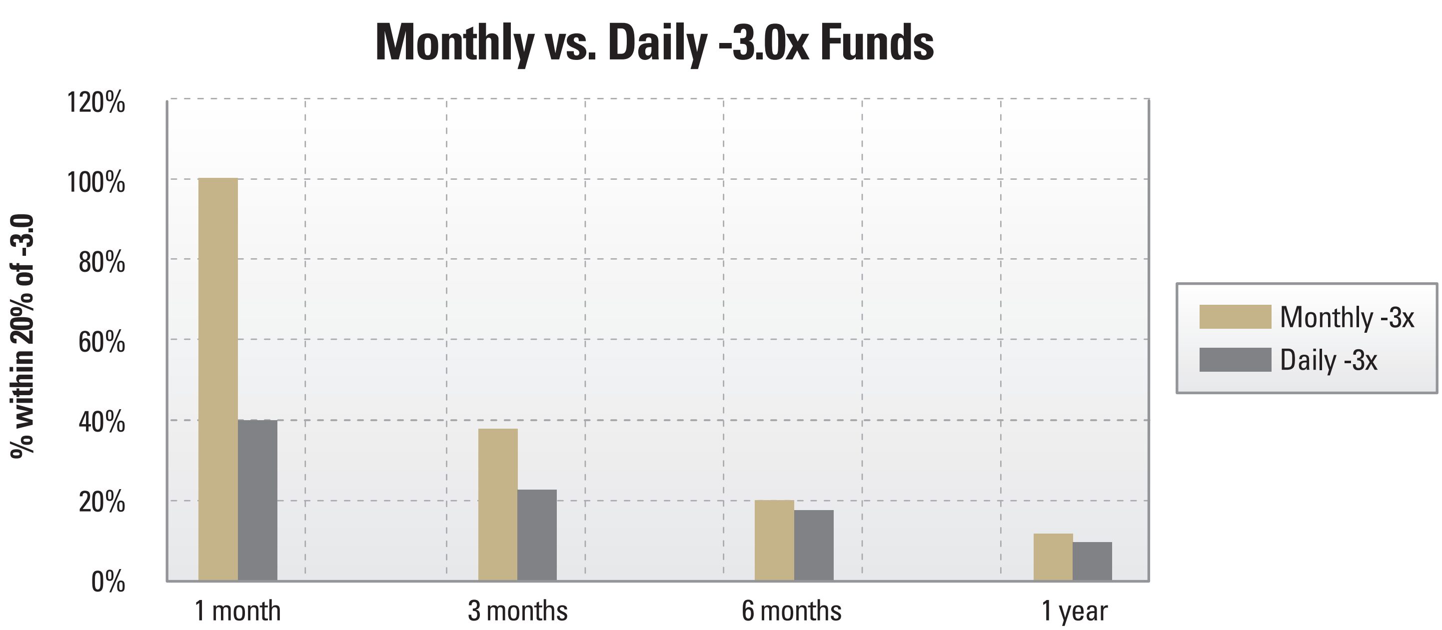


Figure C11: Probability that realized beta will be within 20% of the leverage ratio of -3x for daily and monthly leveraged -3.0x funds, assuming an expected return of 10% and standard deviation of 20%



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